

THE SELBERG INTEGRAL AND YOUNG BOOKS

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ABSTRACT. The Selberg integral is an important integral first evaluated by Selberg in 1944. Stanley found a combinatorial interpretation of the Selberg integral in terms of permutations. In this paper, new combinatorial objects “Young books” are introduced and shown to have a connection with the Selberg integral. This connection gives an enumeration formula for Young books. It is shown that special cases of Young books become standard Young tableaux of various shapes: shifted staircases, squares, certain skew shapes, and certain truncated shapes. As a consequence, product formulas for the number of standard Young tableaux of these shapes are obtained.

1. INTRODUCTION

The Selberg integral is the following integral first evaluated by Selberg [7] in 1944:

$$(1) \quad S_n(\alpha, \beta, \gamma) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{\alpha-1} (1-x_i)^{\beta-1} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2\gamma} dx_1 \cdots dx_n \\ = \prod_{j=1}^n \frac{\Gamma(\alpha + (j-1)\gamma) \Gamma(\beta + (j-1)\gamma) \Gamma(1+j\gamma)}{\Gamma(\alpha + \beta + (n+j-2)\gamma) \Gamma(1+\gamma)},$$

where n is a positive integer and α, β, γ are complex numbers such that $\operatorname{Re}(\alpha) > 0$, $\operatorname{Re}(\beta) > 0$, and $\operatorname{Re}(\gamma) > -\min\{1/n, \operatorname{Re}(\alpha)/(n-1), \operatorname{Re}(\beta)/(n-1)\}$. We refer the reader to Forrester and Warnaar’s exposition [3] for the history and importance of the Selberg integral.

In [10, Exercise 1.10 (b)] Stanley gives a combinatorial interpretation of the Selberg integral when the exponents $\alpha-1, \beta-1$ and 2γ are nonnegative integers by introducing certain permutations. In this paper we define “Selberg books” which are essentially a graphical representation of these permutations as fillings of certain Young diagrams. We then define “Young books” which are special Selberg books. Young books are a generalization of both shifted Young tableaux of staircase shape and standard Young tableaux of square shape. We show that there is a simple relation between the number of Selberg books and the number of Young books by finding generating functions for both objects.

It is well known that the number of standard Young tableaux has a nice product formula due to Frame, Robinson, and Thrall [4] in which every factor is at most the size of the shape. However, the number of standard Young tableaux of a skew shape or a truncated shape may not have such a product formula since it may have a large prime factor compared to the size of the shape. A truncated shape is a diagram obtained from a usual Young diagram in English convention by removing cells in its southwest corner. Standard Young tableaux of truncated shapes were recently considered by Adin and Roichman [2]. They showed that the number of geodesics between two antipodes in the flip graph of triangle-free triangulations is equal to twice the number of standard Young tableaux of certain shifted truncated shape. Adin, King, and Roichman [1] and Panova [5] showed that the number of standard Young tableaux of certain truncated shapes has a product formula. As a consequence of our formula for the Young books, we obtain some product formulas for the number of standard Young tableaux of some skew shapes and truncated shapes.

This paper is organized as follows. In Section 2 we review Stanley’s combinatorial interpretation of the Selberg integral. In Section 3 we define Selberg books and Young books in a simple form

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which are related to the Selberg integral when $\alpha = \beta = 1$. We show that there is a simple relation between their cardinalities by finding generating functions for them. Using this relation and (1) we obtain a formula for the number of Young books. In Section 4 we define Selberg books and Young books in the complete form which are related to the Selberg integral without restriction. Results in Section 2 are extended here. As a consequence we obtain product formulas for the number of standard Young tableaux of a truncated shape obtained from a rectangle by removing a staircase from the southwest corner, and a skew shape obtained by attaching two such truncated shapes. Using generating functions, we find another integral expression for the Selberg integral. In Section 5 we consider generalized Selberg books. We find a product formula for the number of standard Young tableaux of a truncated shape, which is more general than two truncated shapes considered by Panova [5].

2. STANLEY'S COMBINATORIAL INTERPRETATION

In this section we review Stanley's combinatorial interpretation of the Selberg integral in terms of probability when $r = \alpha - 1$, $s = \beta - 1$ and $m = 2\gamma$ are nonnegative integers.

Let $A(n, r, s, m)$ be the following set of letters

$$A(n, r, s, m) = \{x_i : 1 \leq i \leq n\} \cup \{a_{ij}^{(k)} : 1 \leq i < j \leq n, 1 \leq k \leq m\} \\ \cup \{b_i^{(k)} : 1 \leq i \leq n, 1 \leq k \leq r\} \cup \{c_i^{(k)} : 1 \leq i \leq n, 1 \leq k \leq s\}.$$

A permutation of $A(n, r, s, m)$ is called a *Selberg permutation* if the following conditions hold:

- x_1, x_2, \dots, x_n are in this order,
- $a_{ij}^{(k)}$ is between x_i and x_j for $1 \leq i < j \leq n$ and $1 \leq k \leq m$,
- $b_i^{(k)}$ is before x_i for $1 \leq i \leq n$ and $1 \leq k \leq r$, and
- $c_i^{(k)}$ is after x_i for $1 \leq i \leq n$ and $1 \leq k \leq s$.

Let $\text{SP}(n, r, s, m)$ denote the set of Selberg permutations of $A(n, r, s, m)$. For example

$$b_1^{(1)} x_1 a_{13}^{(1)} b_2^{(1)} a_{13}^{(2)} c_1^{(2)} c_1^{(1)} a_{12}^{(2)} a_{12}^{(1)} b_3^{(1)} x_2 a_{23}^{(2)} c_2^{(1)} a_{23}^{(1)} x_3 c_3^{(1)} c_2^{(2)} c_3^{(2)} \in \text{SP}(3, 1, 2, 2).$$

Then we have the following combinatorial interpretation for the Selberg integral, see [10, Exercise 1.10 (b)].

Proposition 2.1. *We have*

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^r (1 - x_i)^s \prod_{1 \leq i < j \leq n} |x_i - x_j|^m dx_1 \cdots dx_n = \frac{n! |\text{SP}(n, r, s, m)|}{((r + s + 1)n + mn(n - 1)/2)!}.$$

For a nonnegative integer n , we define

$$n!! = \prod_{j=0}^{\lfloor (n-1)/2 \rfloor} (n - 2j).$$

In other words,

$$(2k)!! = (2k)(2k - 2) \cdots 2, \quad (2k - 1)!! = (2k - 1)(2k - 3) \cdots 1.$$

By (1), Proposition 2, and the facts $\Gamma(1 + n) = n!$ and $\Gamma(\frac{1}{2} + n) = \frac{(2n-1)!!}{2^n} \sqrt{\pi}$, we obtain the following formula for the number of Selberg permutations.

Proposition 2.2. *We have*

$$|\text{SP}(n, r, s, m)| = \frac{2^n ((r + s + 1)n + mn(n - 1)/2)!}{n!} \prod_{j=1}^n \frac{(jm)!! (2r + (j - 1)m)!! (2s + (j - 1)m)!!}{m!! (2r + 2s + 2 + (n + j - 2)m)!!}.$$

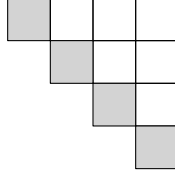
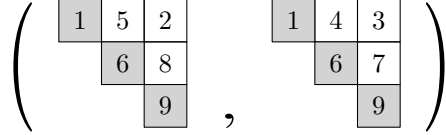


FIGURE 1. A shifted staircase of size 4. The diagonal cells are shaded.

FIGURE 2. A $(3, 2)$ -Selberg book. The diagonal cells are shaded.

3. (n, m) -SELBERG BOOKS AND (n, m) -YOUNG BOOKS

In this section we define (n, m) -Selberg books which are in natural bijection with the Selberg permutations $\text{SB}(n, r, s, m)$ when $r = s = 0$. We then define (n, m) -Young books which are (n, m) -Selberg books with an additional condition. In the next section we will consider more general Selberg books and Young books which are related to $\text{SB}(n, r, s, m)$ for any nonnegative integers r and s .

The *shifted staircase of size n* is the shifted partition $(n, n-1, \dots, 1)$. The cell in the i th row and i th column is called the *i th diagonal cell*. We will identify the shifted staircase of size n with its shifted Young diagram as shown in Figure 1.

Definition 3.1. Let $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$ be shifted staircases of size n . We identify the i th diagonal cells of $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$ for each $1 \leq i \leq n$. We call $\lambda^{(i)}$ the *i th page*. An (n, m) -Selberg book is a filling of the m -tuple $(\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)})$ with integers $1, 2, \dots, n + m \binom{n}{2}$ such that in each page the integer in the i th row and j th column with $i \neq j$ is bigger than the integer in the i th diagonal cell and smaller than the integer in the j th diagonal cell. Let $\text{SB}(n, m)$ be the set of (n, m) -Selberg books.

See Figure 2 for an example of (n, m) -Selberg book.

There is a natural bijection between $\text{SB}(n, m)$ and $\text{SP}(n, 0, 0, m)$ as follows. For $B \in \text{SB}(n, m)$, define the corresponding permutation $\pi = \pi_1 \pi_2 \dots \pi_{n+m \binom{n}{2}}$ by

$$\pi_\ell = \begin{cases} x_i, & \text{if } B \text{ has } \ell \text{ in the } i\text{th diagonal cell,} \\ a_{ij}^{(k)}, & \text{if } B \text{ has } \ell \text{ in the } i\text{th row and } j\text{th column of the } k\text{th shifted staircase with } i \neq j. \end{cases}$$

For instance, the permutation corresponding to the Selberg book in Figure 2 is

$$x_1 a_{13}^{(1)} a_{13}^{(2)} a_{12}^{(2)} a_{12}^{(1)} x_2 a_{23}^{(2)} a_{23}^{(1)} x_3.$$

Thus, by Proposition 2.2 with $r = s = 0$, we obtain a formula for $|\text{SB}(n, m)|$.

Proposition 3.1. *We have*

$$|\text{SB}(n, m)| = \frac{2^n (n + mn(n-1)/2)!}{n! m!^{!n}} \prod_{j=1}^n \frac{((j-1)m)!^{!2} (jm)!}{(2 + (n+j-2)m)!}.$$

Definition 3.2. An (n, m) -Young book is an (n, m) -Selberg book with the additional condition that for each shifted staircase the integers are increasing along each row and column. Let $\text{YB}(n, m)$ be the set of (n, m) -Young books.

Note that an $(n, 1)$ -Young book is a just standard Young tableau of shifted staircase shape. By attaching the two shifted staircases along the diagonal cells after flipping over the second shifted

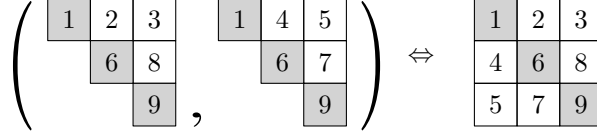


FIGURE 3. The correspondence between $(n, 2)$ -Selberg books and standard Young tableaux of square shape (n^n) . The diagonal cells are shaded.

staircase, an $(n, 2)$ -Young book can be thought of as a standard Young tableau of square shape (n^n) , see Figure 3.

For the rest of this section we will find a simple relation between the cardinalities of $\text{SB}(n, m)$ and $\text{YB}(n, m)$.

We define $\text{SB}(n, m; d_1, \dots, d_{n-1})$ and $\text{YB}(n, m; d_1, \dots, d_{n-1})$ to be, respectively, the set of (n, m) -Selberg books and the set of (n, m) -Young books such that the entries a_1, a_2, \dots, a_n in the diagonal cells satisfy $d_i = a_{i+1} - a_i - 1$ for $i = 1, 2, \dots, n-1$. Note that since we always have $a_1 = 1$ and $a_n = n + m \binom{n}{2}$, the numbers d_1, \dots, d_{n-1} determine a_1, a_2, \dots, a_n , and vice versa.

Proposition 3.2. *For nonnegative integers n and m , we have*

$$\sum_{d_1, \dots, d_{n-1} \geq 0} |\text{SB}(n, m; d_1, \dots, d_{n-1})| \frac{t_1^{d_1} \dots t_{n-1}^{d_{n-1}}}{d_1! \dots d_{n-1}!} = \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \dots + t_{j-1})^m.$$

Proof. We define a *reduced (n, m) -Selberg book* to be a filling of an m -tuple of shifted staircases of size n with integers $1, 2, \dots, n-1$ with repetition allowed such that the diagonal cells are empty and a non-diagonal cell in the i th row and j th column is filled with an integer k satisfying $i \leq k < j$. Let $\text{RSB}(n, m; d_1, \dots, d_{n-1})$ denote the set of reduced (n, m) -Selberg books with d_1 1's, d_2 2's, and so on. By definition we have

$$\sum_{d_1, \dots, d_{n-1} \geq 0} |\text{RSB}(n, m; d_1, \dots, d_{n-1})| t_1^{d_1} \dots t_{n-1}^{d_{n-1}} = \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \dots + t_{j-1})^m.$$

Let $B \in \text{SB}(n, m; d_1, \dots, d_{n-1})$. Then the entries a_1, \dots, a_n in the diagonal cells of B satisfy $d_i = a_{i+1} - a_i - 1$. Let B' be the reduced (n, m) -Selberg book obtained from B by replacing the d_i integers $a_i + 1, a_i + 2, \dots, a_{i+1} - 1$ in B with i 's for each $i = 1, 2, \dots, n-1$. It is easy to see that the map $B \mapsto B'$ is 1-to- $d_1! \dots d_{n-1}!$, which finishes the proof. \square

By computing the volume of the Gelfand-Tsetlin polytopes in two different ways, Postnikov [6] showed that

$$(2) \quad \sum_{d_1, \dots, d_{n-1} \geq 0} |\text{YB}(n, 1; d_1, \dots, d_{n-1})| \frac{t_1^{d_1} \dots t_{n-1}^{d_{n-1}}}{d_1! \dots d_{n-1}!} = \prod_{1 \leq i < j \leq n} \frac{t_i + t_{i+1} + \dots + t_{j-1}}{j - i}.$$

Theorem 3.3. *We have*

$$(3) \quad |\text{SB}(n, m; d_1, \dots, d_{n-1})| = (1!2! \dots (n-1)!)^m \cdot |\text{YB}(n, m; d_1, \dots, d_{n-1})|,$$

$$(4) \quad |\text{SB}(n, m)| = (1!2! \dots (n-1)!)^m \cdot |\text{YB}(n, m)|.$$

Proof. Since (4) is obtained from (3) by summing over all d_1, \dots, d_{n-1} , it suffices to prove (3). By Proposition 3.2 and (2), we have

$$(5) \quad |\text{SB}(n, 1; d_1, \dots, d_{n-1})| = 1!2! \dots (n-1)! \cdot |\text{YB}(n, 1; d_1, \dots, d_{n-1})|.$$

Hence, the theorem is true for the case $m = 1$. We now consider for an arbitrary m .

Let a_1, a_2, \dots, a_n be the integers satisfying $a_0 = 1$ and $d_i = a_{i+1} - a_i - 1$ for $i = 1, 2, \dots, n-1$. For a set X of $\binom{n}{2}$ integers, let $\text{SB}_X(n, 1; d_1, \dots, d_{n-1})$ be the set of fillings of a shifted staircase of size n with integers in $X \cup \{a_1, \dots, a_n\}$ so that the i th diagonal cell is filled with a_i and a

non-diagonal cell in the i th row and j th column is filled with an integer k satisfying $a_i < k < a_j$. By considering each shifted staircase separately we get

$$(6) \quad |\text{SB}(n, m; d_1, \dots, d_{n-1})| = \sum_{X_1, \dots, X_m} \prod_{i=1}^m |\text{SB}_{X_i}(n, 1; d_1, \dots, d_{n-1})|,$$

where the sum is over all subsets X_1, \dots, X_m of $\{1, 2, \dots, n + m \binom{n}{2}\} \setminus \{a_1, \dots, a_n\}$ such that $|X_i| = \binom{n}{2}$ for all i , and

$$X_1 \cup \dots \cup X_m = \left\{1, 2, \dots, n + m \binom{n}{2}\right\} \setminus \{a_1, \dots, a_n\}.$$

Similarly we can define $\text{YB}_X(n, 1; d_1, \dots, d_{n-1})$ and obtain

$$(7) \quad |\text{YB}(n, m; d_1, \dots, d_{n-1})| = \sum_{X_1, \dots, X_m} \prod_{i=1}^m |\text{YB}_{X_i}(n, 1; d_1, \dots, d_{n-1})|.$$

For given X_i , we have

$$|\text{SB}_{X_i}(n, 1; d_1, \dots, d_{n-1})| = |\text{SB}(n, 1; d'_1, \dots, d'_{n-1})|,$$

$$|\text{YB}_{X_i}(n, 1; d_1, \dots, d_{n-1})| = |\text{YB}(n, 1; d'_1, \dots, d'_{n-1})|,$$

for the same d'_1, \dots, d'_{n-1} . Thus by (5) we have

$$|\text{SB}_{X_i}(n, 1; d_1, \dots, d_{n-1})| = 1!2! \cdots (n-1)! \cdot |\text{YB}_{X_i}(n, 1; d_1, \dots, d_{n-1})|.$$

Applying the above equation to (6) and (7) we get (3). \square

By Proposition 3.2 and (3) we obtain the following generalization of Postnikov's result (2).

Corollary 3.4. *We have*

$$\sum_{d_1, \dots, d_{n-1} \geq 0} |\text{YB}(n, m; d_1, \dots, d_{n-1})| \frac{t_1^{d_1} \cdots t_{n-1}^{d_{n-1}}}{d_1! \cdots d_{n-1}!} = \left(\prod_{1 \leq i < j \leq n} \frac{t_i + t_{i+1} + \cdots + t_{j-1}}{j - i} \right)^m.$$

We note that Corollary 3.4 can also be proved directly from (2) using (7).

By Proposition 3.1 and (4) we get the number of (n, m) -Young books.

Corollary 3.5. *We have*

$$|\text{YB}(n, m)| = \frac{2^n (n + mn(n-1)/2)!}{n! m!!^n} \prod_{j=1}^n \frac{((j-1)m)!!^2 (jm)!!}{(j-1)!^m (2 + (n+j-2)m)!!}.$$

If $m = 1$ in Corollary 3.5, then we get the hook length formula for the number of standard Young tableaux of shifted staircase shape of size n . If $m = 2$ in Corollary 3.5, then we get the hook length formula for the number of standard Young tableaux of square shape (n^n) . This gives a semi-combinatorial proof of the Selberg integral for $\alpha = \beta = 1$ and $\gamma \in \{1/2, 1\}$.

We can obtain a combinatorial proof of the Selberg integral formula when $r = \alpha - 1, s = \beta - 1$ and $m = 2\gamma$ are nonnegative integers if we solve the following two problems.

Problem 3.1. Find a combinatorial proof of Theorem 3.3.

Problem 3.2. Find a combinatorial proof of Corollary 3.5.

One can consider Young books of shape $(\lambda^{(1)}, \dots, \lambda^{(m)})$ for shifted Young diagrams $\lambda^{(i)}$ with the same number of rows. However, in this case we do not seem to have a nice product formula. For instance, the number of Young books of shape

$$((6, 2, 1), (5, 4, 1), (5, 2, 1), (4, 2, 1))$$

is equal to

$$2^4 \cdot 3 \cdot 5^2 \cdot 7 \cdot 17 \cdot 19 \cdot 23 \cdot 1649819.$$

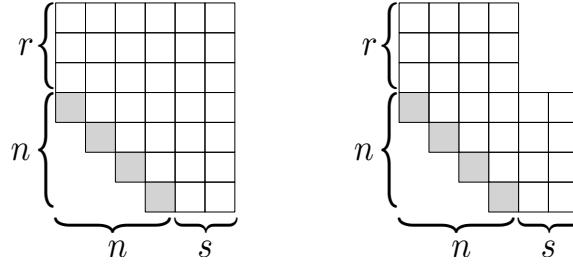


FIGURE 4. An (n, r, s) -staircase on the left and an $(n, r, s)^-$ -staircase on the right. The diagonal cells are shaded.

4. $(n, \mathbf{r}, \mathbf{s})$ -SELBERG BOOKS AND $(n, \mathbf{r}, \mathbf{s})$ -YOUNG BOOKS

For a nonnegative integer r , a *composition* of r is a sequence $\mathbf{r} = (r_1, r_2, \dots, r_m)$ of nonnegative integers summing to r . In this case we write $\mathbf{r} \models r$ and say that the *length* of \mathbf{r} is m .

In this section, for nonnegative integers n, r, s, m and compositions $\mathbf{r} \models r$ and $\mathbf{s} \models s$ of length m , we define $(n, \mathbf{r}, \mathbf{s})$ -Selberg books and $(n, \mathbf{r}, \mathbf{s})$ -Young books which are related to $\text{SB}(n, r, s, m)$.

An (n, r, s) -*staircase* is the diagram obtained from an $(r+n) \times (n+s)$ rectangle by removing the cells below the diagonal cells, where the cell in the $(i+r)$ th row and i th column is called the i th diagonal cell. An $(n, r, s)^-$ -*staircase* is the diagram obtained from an (n, r, s) -staircase by removing the $r \times s$ rectangle at the northeast corner. See Figure 4.

Definition 4.1. Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. For $1 \leq i \leq m$, let $\lambda^{(i)}$ be a $(n, r_i, s_i)^-$ -staircase. We identify the i th diagonal cells of $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$ for each $1 \leq i \leq n$. We call $\lambda^{(i)}$ the i th *page*. A $(n, \mathbf{r}, \mathbf{s})^-$ -*Selberg book* is a filling of $(\lambda^{(1)}, \dots, \lambda^{(m)})$ with $1, 2, \dots, (r+s+1)n + m \binom{n}{2}$ such that the integer in a non-diagonal cell of each page is bigger than the integer in the diagonal cell of the same row and smaller than the integer in the diagonal cell of the same column. Let $\text{SB}^-(n, \mathbf{r}, \mathbf{s})$ denote the set of $(n, \mathbf{r}, \mathbf{s})^-$ -Selberg books.

The bijection between $\text{SB}(n, m)$ and $\text{SP}(n, 0, 0, m)$ in Section 3 can easily be extended to a bijection between $\text{SB}^-(n, \mathbf{r}, \mathbf{s})$ and $\text{SP}(n, r, s, m)$. Notice that the cardinality of $\text{SB}^-(n, \mathbf{r}, \mathbf{s})$ depends only on n, r, s . Thus we obtain the following proposition.

Proposition 4.1. Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. Then

$$|\text{SB}^-(n, \mathbf{r}, \mathbf{s})| = |\text{SP}(n, r, s, m)|.$$

Now we define $(n, \mathbf{r}, \mathbf{s})$ -Selberg books.

Definition 4.2. Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. For $1 \leq i \leq m$, let $\lambda^{(i)}$ be an (n, r_i, s_i) -staircase. We identify the i th diagonal cells of $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$ for each $1 \leq i \leq n$. An $(n, \mathbf{r}, \mathbf{s})$ -*Selberg book* is a filling of $(\lambda^{(1)}, \dots, \lambda^{(m)})$ with $1, 2, \dots, (r+s+1)n + m \binom{n}{2} + \sum_{i=1}^m r_i s_i$ such that in each page the integer in a non-diagonal cell is bigger than the integer in the diagonal cell of the same row and smaller than the integer in the diagonal cell of the same column. Let $\text{SB}(n, \mathbf{r}, \mathbf{s})$ denote the set of $(n, \mathbf{r}, \mathbf{s})$ -Selberg books.

There is a simple relation between $|\text{SB}(n, \mathbf{r}, \mathbf{s})|$ and $|\text{SB}^-(n, \mathbf{r}, \mathbf{s})|$.

Proposition 4.2. Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. Then

$$|\text{SB}(n, \mathbf{r}, \mathbf{s})| = |\text{SB}^-(n, \mathbf{r}, \mathbf{s})| \frac{((r+s+1)n + m \binom{n}{2} + \sum_{i=1}^m r_i s_i)!}{((r+s+1)n + m \binom{n}{2})!}.$$

Proof. This follows from the observation that there are no restrictions on the entries of the cells in $(n, \mathbf{r}, \mathbf{s})$ -Selberg books which are not in $(n, \mathbf{r}, \mathbf{s})^-$ -Selberg books. \square

Definition 4.3. For $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$, we define an $(n, \mathbf{r}, \mathbf{s})$ -*Young book* to be an $(n, \mathbf{r}, \mathbf{s})$ -Selberg book such that in each page the entries are increasing from left to right and from top to bottom. Let $\text{YB}(n, \mathbf{r}, \mathbf{s})$ denote the set of $(n, \mathbf{r}, \mathbf{s})$ -Young books. We

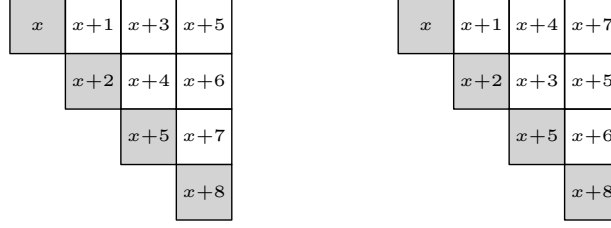


FIGURE 5. The diagram on the left shows the typical form of the entries in rows $k+1, k+2, \dots, k+\ell$ and columns $k+1, k+2, \dots, k+\ell$ of $B \in \text{YB}(n, 1; d_1, d_2, \dots, d_{n-1})$ when $d_{k+1} = 1, d_{k+2} = 2, \dots, d_{k+\ell-1} = \ell - 1$. The diagram on the right shows that, in the case of $B \in \text{SB}(n, 1; d_1, d_2, \dots, d_{n-1})$, for $1 \leq j \leq \ell$, the non-diagonal entries in column $k+j$ and below row k are obtained by permuting those in the same cells of the diagram on the left. In this example, we have $\ell = 4$.

also define $\text{SB}(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)$ and $\text{YB}(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)$ to be, respectively, the set of $(n, \mathbf{r}, \mathbf{s})$ -Selberg books and the set of $(n, \mathbf{r}, \mathbf{s})$ -Young books whose diagonal entries a_1, \dots, a_n satisfy $d_i = a_{i+1} - a_i - 1$ for $i = 0, 1, 2, \dots, n$, where $a_0 = 1$ and $a_{n+1} = (r+s+1)n + m \binom{n}{2} + \sum_{i=1}^m r_i s_i + 1$.

The following lemma is an immediate consequence of the definition of Selberg books and Young books.

Lemma 4.3. *Suppose that d_1, d_2, \dots, d_{n-1} is a sequence of nonnegative integers such that $d_{k+1} = 1, d_{k+2} = 2, \dots, d_{k+\ell-1} = \ell - 1$ for some $k, \ell \geq 0$. Then, for any $B \in \text{YB}(n, 1; d_1, d_2, \dots, d_{n-1})$, the entries in rows $k+1, k+2, \dots, k+\ell$ and columns $k+1, k+2, \dots, k+\ell$ are completely determined by d_1, \dots, d_{n-1} . More precisely, for $1 \leq i, j \leq \ell$, if x is the entry in the $(k+1)$ st diagonal cell, which is determined by d_1, \dots, d_{n-1} , then the entry in row $k+i$ and column $k+j$ is $x + \binom{j-1}{2} + i$.*

Moreover, if $B \in \text{SB}(n, 1; d_1, d_2, \dots, d_{n-1})$ and x is the entry in the $(k+1)$ st diagonal cell, then the entries in column $k+j$ and in rows $k+1, k+2, \dots, k+j-1$ form a permutation of $x + \binom{j-1}{2} + 1, x + \binom{j-1}{2} + 2, \dots, x + \binom{j-1}{2} + j - 1$.

Figure 5 illustrates the situation in Lemma 4.3.

Proposition 4.4. *Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. Then we have*

$$|\text{SB}(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| = |\text{YB}(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| \prod_{i=1}^m \frac{1!2! \cdots (n+r_i+s_i-1)!}{1!2! \cdots (r_i-1)!1!2! \cdots (s_i-1)!}.$$

Proof. We will prove this only for the case $m = 1$. For $m \geq 2$, we can use the same idea as in the proof of Theorem 3.3. Let $m = 1, \mathbf{r} = (r)$, and $\mathbf{s} = (s)$. Then by Lemma 4.3 we have

$$|\text{SB}(n, \mathbf{r}, \mathbf{s}; d_0, \dots, d_n)| = |\text{SB}(n+r+s, 1; 1, 2, \dots, r-1, d_0, \dots, d_n, 1, 2, \dots, s-1)| \\ \times 1!2! \cdots (r-1)!1!2! \cdots (s-1)!,$$

$$|\text{YB}(n, \mathbf{r}, \mathbf{s}; d_0, \dots, d_n)| = |\text{YB}(n+r+s, 1; 1, 2, \dots, r-1, d_0, \dots, d_n, 1, 2, \dots, s-1)|.$$

By the above equations and (3), we get the desired formula for the case $m = 1$. \square

By Propositions 2.2, 4.1, 4.2, and 4.4, we get a formula for $|\text{YB}(n, \mathbf{r}, \mathbf{s})|$.

Theorem 4.5. *Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. Then*

$$|\text{YB}(n, \mathbf{r}, \mathbf{s})| = \left((r+s+1)n + m \binom{n}{2} + \sum_{i=1}^m r_i s_i \right)! \prod_{i=1}^m \frac{1!2! \cdots (r_i-1)!1!2! \cdots (s_i-1)!}{1!2! \cdots (n+r_i+s_i-1)!} \\ \times \frac{2^n}{n!} \prod_{j=1}^n \frac{(jm)!!(2r+(j-1)m)!!(2s+(j-1)m)!!}{m!!(2r+2s+2+(n+j-2)m)!!}.$$

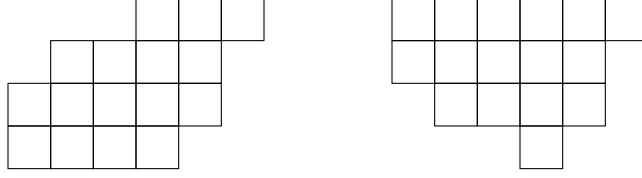


FIGURE 6. The skew shape λ/μ on the left and the truncated shape $\lambda \setminus \mu$ on the right for $\lambda = (6, 5, 5, 4)$ and $\mu = (3, 1)$.

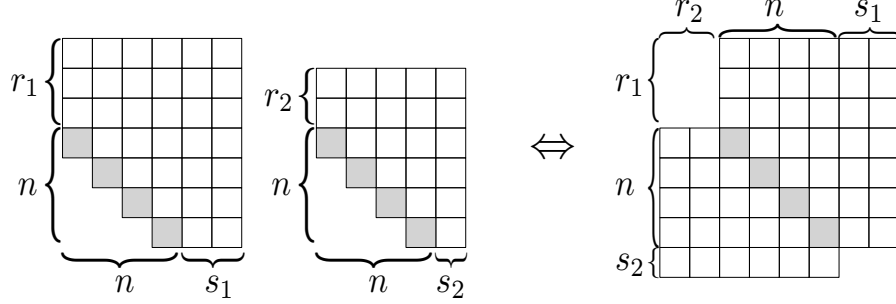


FIGURE 7. The skew shape λ/μ on the right is obtained by attaching an (n, r_1, r_2) -staircase and an (n, r_2, s_2) -staircase along the diagonal cells. The diagonal cells are shaded and the (n, r_2, s_2) -staircase is flipped when attached.

For two partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_\ell)$, the *skew shape* λ/μ is defined to be the set-theoretic difference $\lambda - \mu$ of their Young diagrams. We define the *truncated shape* $\lambda \setminus \mu$ to be the diagram obtained from the Young diagram of λ by removing the μ_i cells from the left in the $(k + 1 - i)$ th row for $i = 1, 2, \dots, \ell$. See Figure 6.

Notice that, when $m = 1$, an $(n, (r), (s))$ -Young book is the same as a standard Young tableau of truncated shape $\lambda \setminus \mu$ for $\lambda = ((n + s)^{r+n})$ and $\mu = (n - 1, n - 2, \dots, 1)$. In this case we obtain the following corollary from Theorem 4.5.

Corollary 4.6. *The number of standard Young tableaux of truncated shape*

$$((n + s)^{r+n}) \setminus (n - 1, n - 2, \dots, 1)$$

is

$$\left((r + s + 1)n + \binom{n}{2} + rs \right)! \frac{2^n F(r) F(s)}{n! F(n + r + s)} \prod_{j=1}^n \frac{(j)!! (2r + j - 1)!! (2s + j - 1)!!}{(2r + 2s + n + j)!!},$$

where $F(k) = 1!2! \dots (k - 1)!$.

Panova [5] also found a product formula for the number in the above corollary. In the next section we find a product formula for the number of standard Young tableaux of a more general shape.

When $m = 2$, by attaching the two pages along the diagonal cells, an $(n, (r_1, r_2), (s_1, s_2))$ -Young book can be thought of as a standard Young tableau of skew shape λ/μ for

$$(8) \quad \lambda = ((r_2 + n + s_1)^{r_1 + n}, (r_2 + n)^{s_2}), \quad \mu = (r_2^{r_1}).$$

See Figure 7 for such a construction.

Corollary 4.7. *Let λ and μ be the partitions given in (8) whose diagram is drawn on the right in Figure 7. Then the number of standard Young tableaux of skew shape λ/μ is*

$$\frac{2^n ((r + s)n + n^2 + r_1 s_1 + r_2 s_2)! F(r_1) F(r_2) F(s_1) F(s_2)}{n! F(n + r_1 + s_1) F(n + r_2 + s_2)} \prod_{j=1}^n \frac{(2j)!! (2r + 2j - 2)!! (2s + 2j - 2)!!}{(2r + 2s + 2n + 2j - 2)!!}.$$

where $F(k) = 1!2! \dots (k - 1)!$.

Notice that the skew shape λ/μ in Corollary 4.7 is obtained from a rectangle by removing a smaller rectangle both from its northwest corner and southeast corner. One may ask if there is a product formula for the number of standard Young tableaux of any skew shape obtained in this way. If $\lambda = (7, 7, 7, 7, 5, 5)$ and $\mu = (4, 4)$, then the number of standard Young tableaux of λ/μ has a factor of 9173. Thus, in general, we cannot expect a product formula for the number of standard Young tableaux of such a skew shape.

There is a formula for the number of standard Young tableaux of skew shape as a determinant, see [9, 7.16.3 Corollary]. It would be interesting to prove Corollary 4.7 using the determinantal formula.

By the same arguments as in the previous section, one can prove the following two propositions.

Proposition 4.8. *Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. Then we have*

$$\begin{aligned} \sum_{d_0, d_1, \dots, d_n \geq 0} |\text{SB}(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| \frac{t_0^{d_0} t_1^{d_1} \dots t_n^{d_n}}{d_0! d_1! \dots d_n!} \\ = \prod_{i=1}^n (t_0 + t_1 + \dots + t_{i-1})^r (t_i + t_{i+1} + \dots + t_n)^s \\ \times \prod_{i=1}^m (t_0 + t_1 + \dots + t_n)^{r_i s_i} \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \dots + t_{j-1})^m. \end{aligned}$$

Proposition 4.9. *Let $\mathbf{r} = (r_1, \dots, r_m) \models r$ and $\mathbf{s} = (s_1, \dots, s_m) \models s$. Then we have*

$$\begin{aligned} \sum_{d_0, d_1, \dots, d_n \geq 0} |\text{YB}(n, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| \frac{t_0^{d_0} t_1^{d_1} \dots t_n^{d_n}}{d_0! d_1! \dots d_n!} \\ = \prod_{i=1}^n (t_0 + t_1 + \dots + t_{i-1})^r (t_i + t_{i+1} + \dots + t_n)^s \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \dots + t_{j-1})^m \\ \times \prod_{i=1}^m (t_0 + t_1 + \dots + t_n)^{r_i s_i} \frac{1!2! \dots (r_i - 1)! 1!2! \dots (s_i - 1)!}{1!2! \dots (n + r_i + s_i - 1)!}. \end{aligned}$$

Using Proposition 4.8 we can obtain another integral expression for the Selberg integral. First, note that

$$\int_0^\infty x^n e^{-x} dx = n!.$$

Thus

$$|\text{SB}(n, r, s, m)| = \sum_{d_0, d_1, \dots, d_n \geq 0} |\text{SB}(n, r, s, m; d_0, d_1, \dots, d_n)|$$

is equal to

$$\int_0^\infty \dots \int_0^\infty \sum_{d_0, d_1, \dots, d_n \geq 0} |\text{SB}(n, r, s, m; d_0, d_1, \dots, d_n)| \frac{t_0^{d_0} t_1^{d_1} \dots t_n^{d_n}}{d_0! d_1! \dots d_n!} e^{-t_0 - t_1 - \dots - t_n} dt_0 dt_1 \dots dt_n.$$

Using Propositions 2.1 and 4.8 we get the following.

Proposition 4.10. *We have*

$$\begin{aligned} \int_0^\infty \dots \int_0^\infty \prod_{i=1}^n (t_0 + t_1 + \dots + t_{i-1})^r (t_i + t_{i+1} + \dots + t_n)^s \\ \times \prod_{1 \leq i < j \leq n} (t_i + t_{i+1} + \dots + t_{j-1})^m e^{-t_0 - t_1 - \dots - t_n} dt_0 dt_1 \dots dt_n \\ = \frac{((r + s + 1)n + mn(n - 1)/2)!}{n!} \int_0^1 \dots \int_0^1 \prod_{i=1}^n x_i^r (1 - x_i)^s \prod_{1 \leq i < j \leq n} |x_i - x_j|^m dx_1 \dots dx_n. \end{aligned}$$

We note that it is also possible to prove the above proposition using the change of variables.

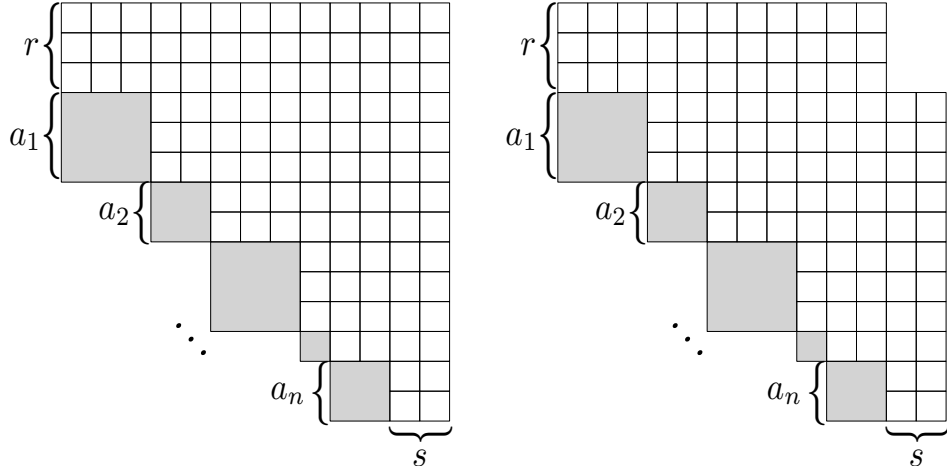


FIGURE 8. An (\mathbf{a}, r, s) -staircase on the left and an $(\mathbf{a}, r, s)^-$ -staircase on the right. The diagonal cells are shaded.

4	2	1	12	3
5	8	7	6	15
		9	10	14
			13	11

1	2	3	5	9
4	6	7	8	12
		10	11	13
			14	15

FIGURE 9. An $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Selberg book on the left and an $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Young book on the right, where $\mathbf{a} = (1, 2), \mathbf{r} = (1), \mathbf{s} = (2)$. The diagonal cells are shaded.

5. GENERALIZED SELBERG BOOKS AND YOUNG BOOKS

In this section we generalize Selberg books and Young books so that a diagonal cell can be a bigger square.

Let $\mathbf{a} = (a_1, \dots, a_n) \models a$. An (\mathbf{a}, r, s) -staircase is the diagram obtained from the truncated shape $\lambda \setminus \mu$ by merging the cells in rows

$$(9) \quad r + a_1 + \dots + a_{i-1} + 1, r + a_1 + \dots + a_{i-1} + 2, \dots, r + a_1 + \dots + a_{i-1} + a_i,$$

and columns

$$(10) \quad a_1 + \dots + a_{i-1} + 1, a_1 + \dots + a_{i-1} + 2, \dots, a_1 + \dots + a_{i-1} + a_i,$$

into a single cell, called the i th diagonal cell, where

$$\lambda = ((a + s)^{(r+a)}), \quad \mu = ((a_1 + \dots + a_{n-1})^{a_n}, (a_1 + \dots + a_{n-2})^{a_{n-1}}, \dots, a_1^{a_2}).$$

We will consider that the i th diagonal cell is contained in every row whose row index is in (9), and in every column whose column index is in (10). An $(\mathbf{a}, r, s)^-$ -staircase is the diagram obtained from an (\mathbf{a}, r, s) -staircase by removing the $r \times s$ rectangle in the northeast corner. See Figure 8.

Throughout this section we will use the following notation. Let $\mathbf{a} = (a_1, \dots, a_n) \models a$, $\mathbf{r} = (r_1, \dots, r_m) \models r$, $\mathbf{s} = (s_1, \dots, s_m) \models s$, and

$$N = n + a(r + s) + m \sum_{1 \leq i < j \leq n} a_i a_j + \sum_{i=1}^m r_i s_i,$$

$$N^- = n + a(r + s) + m \sum_{1 \leq i < j \leq n} a_i a_j.$$

Definition 5.1. For $1 \leq i \leq m$, let $\lambda^{(i)}$ be an (\mathbf{a}, r_i, s_i) -staircase. We identify the i th diagonal cells of $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}$ for all $1 \leq i \leq n$. An $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Selberg book is a filling of $(\lambda^{(1)}, \dots, \lambda^{(m)})$ with $1, 2, \dots, N$ such that the integer in a non-diagonal cell is bigger than the integer in the diagonal cell of the same row and smaller than the integer in the diagonal cell of the same column. See Figure 9. Let $\text{SB}(\mathbf{a}, \mathbf{r}, \mathbf{s})$ denote the set of $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Selberg books.

Now, for $1 \leq i \leq m$, let $\mu^{(i)}$ be an $(\mathbf{a}, r_i, s_i)^-$ -staircase. We identify the i th diagonal cell of $\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(m)}$ for each $1 \leq i \leq n$. An $(\mathbf{a}, \mathbf{r}, \mathbf{s})^-$ -Selberg book is a filling of $(\mu^{(1)}, \dots, \mu^{(m)})$ with $1, 2, \dots, N^-$ such that the integer in a non-diagonal cell is bigger than the integer in the diagonal cell of the same row and smaller than the integer in the diagonal cell of the same column. Let $\text{SB}^-(\mathbf{a}, \mathbf{r}, \mathbf{s})$ denote the set of $(\mathbf{a}, \mathbf{r}, \mathbf{s})^-$ -Selberg books.

There is a simple relation between $|\text{SB}(\mathbf{a}, \mathbf{r}, \mathbf{s})|$ and $|\text{SB}^-(\mathbf{a}, \mathbf{r}, \mathbf{s})|$.

Proposition 5.1. *We have*

$$|\text{SB}(\mathbf{a}, \mathbf{r}, \mathbf{s})| = |\text{SB}^-(\mathbf{a}, \mathbf{r}, \mathbf{s})| \frac{N!}{(N^-)!}.$$

Proof. The proof is similar to that of Proposition 4.2. □

By the same idea as in Proposition 2.1, we obtain the following Proposition.

Proposition 5.2. *We have*

$$\frac{n!}{(N^-)!} |\text{SB}^-(\mathbf{a}, \mathbf{r}, \mathbf{s})| = \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{r_i a_i} (1 - x_i)^{s_i a_i} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{m a_i a_j} dx_1 \cdots dx_n.$$

Definition 5.2. We define an $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Young book to be an $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Selberg book such that in each page entries are increasing from left to right in each row and from top to bottom in each column. See Figure 9. Let $\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s})$ denote the set of $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Young books. We also define $\text{SB}(\mathbf{a}, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)$ and $\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)$ to be, respectively, the set of $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Selberg books and the set of $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Young books whose diagonal entries a_1, \dots, a_n satisfy $d_i = a_{i+1} - a_i - 1$ for $i = 0, 1, 2, \dots, n$, where $a_0 = 1$ and $a_{n+1} = N + 1$.

There is a simple relation between $|\text{SB}(\mathbf{a}, \mathbf{r}, \mathbf{s})|$ and $|\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s})|$.

Proposition 5.3. *We have*

$$|\text{SB}(\mathbf{a}, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| = |\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s}; d_0, d_1, \dots, d_n)| \prod_{i=1}^m \frac{F(a + r_i + s_i)}{F(a_1) \cdots F(a_n) F(r_i) F(s_i)},$$

where $F(k) = 1!2! \cdots (k-1)!$.

Proof. The proof is similar to that of Proposition 4.4. We will prove this only for the case $m = 1$. For $m \geq 2$, we can use the same idea as in the proof of Theorem 3.3.

Let $m = 1, \mathbf{r} = (r), \mathbf{s} = (s)$. By Lemma 4.3, we have

$$\begin{aligned} |\text{SB}(\mathbf{a}, (r), (s); d_0, \dots, d_n)| \\ = |\text{SB}(a + r + s, 1; 1, 2, \dots, r - 1, d_0, 1, 2, \dots, a_1 - 1, d_1, 1, 2, \dots, a_2 - 1, \dots, \\ d_{n-1}, 1, 2, \dots, a_n - 1, d_n, 1, 2, \dots, s - 1)| \cdot F(r) F(a_1) F(a_2) \cdots F(a_n) F(s), \end{aligned}$$

$$\begin{aligned} |\text{YB}(\mathbf{a}, (r), (s); d_0, \dots, d_n)| \\ = |\text{YB}(a + r + s, 1; 1, 2, \dots, r - 1, d_0, 1, 2, \dots, a_1 - 1, d_1, 1, 2, \dots, a_2 - 1, \dots, \\ d_{n-1}, 1, 2, \dots, a_n - 1, d_n, 1, 2, \dots, s - 1)|. \end{aligned}$$

By the above equations and (3), we get the desired formula for the case $m = 1$. □

If $\mathbf{a} = (k^n)$, then we can evaluate $|\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s})|$.

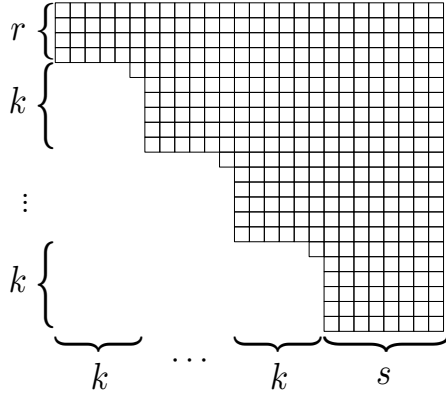


FIGURE 10. The truncated shape $\lambda \setminus \mu$ for $\lambda = ((kn + s)^{r+kn})$ and $\mu = ((kn)^{k-1}, kn - 1, (kn - k)^{k-1}, kn - k - 1, \dots, k^{k-1}, k - 1)$.

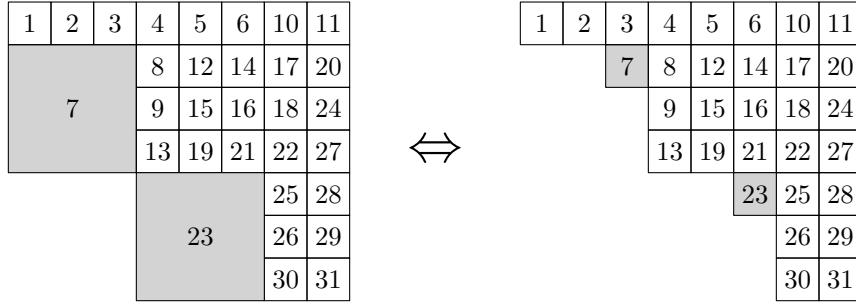


FIGURE 11. The correspondance between $((k^n), (r), (s))$ -Young books and standard Young tableaux of truncated shape in Figure 10.

Corollary 5.4. *Let $\mathbf{a} = (k^n)$. Then*

$$|\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s})| = \frac{2^n ((kr + ks + 1)n + k^2 m \binom{n}{2} + \sum_{i=1}^m r_i s_i)!}{n!} \prod_{i=1}^m \frac{F(k)^n F(r_i) F(s_i)}{F(kn + r_i + s_i)} \\ \times \prod_{j=1}^n \frac{(jk^2 m)!! (2kr + (j-1)k^2 m)!! (2ks + (j-1)k^2 m)!!}{(k^2 m)!! (2kr + 2ks + 2 + (n+j-2)k^2 m)!!}.$$

Proof. By Propositions 5.1, 5.2, and 5.3, we have

$$|\text{YB}(\mathbf{a}, \mathbf{r}, \mathbf{s})| = \frac{((kr + ks + 1)n + k^2 m \binom{n}{2} + \sum_{i=1}^m r_i s_i)!}{n!} \prod_{i=1}^m \frac{F(a_1) \cdots F(a_n) F(r_i) F(s_i)}{F(a + r_i + s_i)} \\ \times \int_0^1 \cdots \int_0^1 \prod_{i=1}^n x_i^{kr} (1 - x_i)^{ks} \prod_{1 \leq i < j \leq n} |x_i - x_j|^{k^2 m} dx_1 \cdots dx_n.$$

We can now use the Selberg integral formula (1) with $\alpha = kr + 1$, $\beta = ks + 1$, and $\gamma = 2k^2 m$, which finishes the proof. \square

If $\mathbf{a} = (k^n)$, $\mathbf{r} = (r)$, $\mathbf{s} = (s)$, then by replacing the each diagonal cell by a 1×1 cell located at the northeast corner of the diagonal cell, we can consider an $(\mathbf{a}, \mathbf{r}, \mathbf{s})$ -Young book as a standard Young tableau of truncated shape $\lambda \setminus \mu$ shown in Figure 10. See Figure 11 for the illustration of this correspondence. Thus we get the following corollary.

Corollary 5.5. *The number of standard Young tableaux of truncated shape in Figure 10 is equal to*

$$\frac{2^n \left((kr + ks + 1)n + k^2 \binom{n}{2} + rs \right)!}{n!} \frac{F(k)^n F(r) F(s)}{F(kn + r + s)} \times \prod_{j=1}^n \frac{(jk^2)!! (2kr + (j-1)k^2)!! (2ks + (j-1)k^2)!!}{(k^2)!! (2kr + 2ks + 2 + (n+j-2)k^2)!!},$$

where $F(n) = 1!2! \dots (n-1)!$.

Panova [5] found a formula for the number of standard Young tableaux of truncated shape $(n^m) \setminus (k-1, k-2, \dots, 1)$ and $(n^m) \setminus (k^{k-1}, k-1)$. Both of these truncated shapes are special cases of the truncated shape in Corollary 5.5.

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